MTR08114 Robotics Inverse Kinematics

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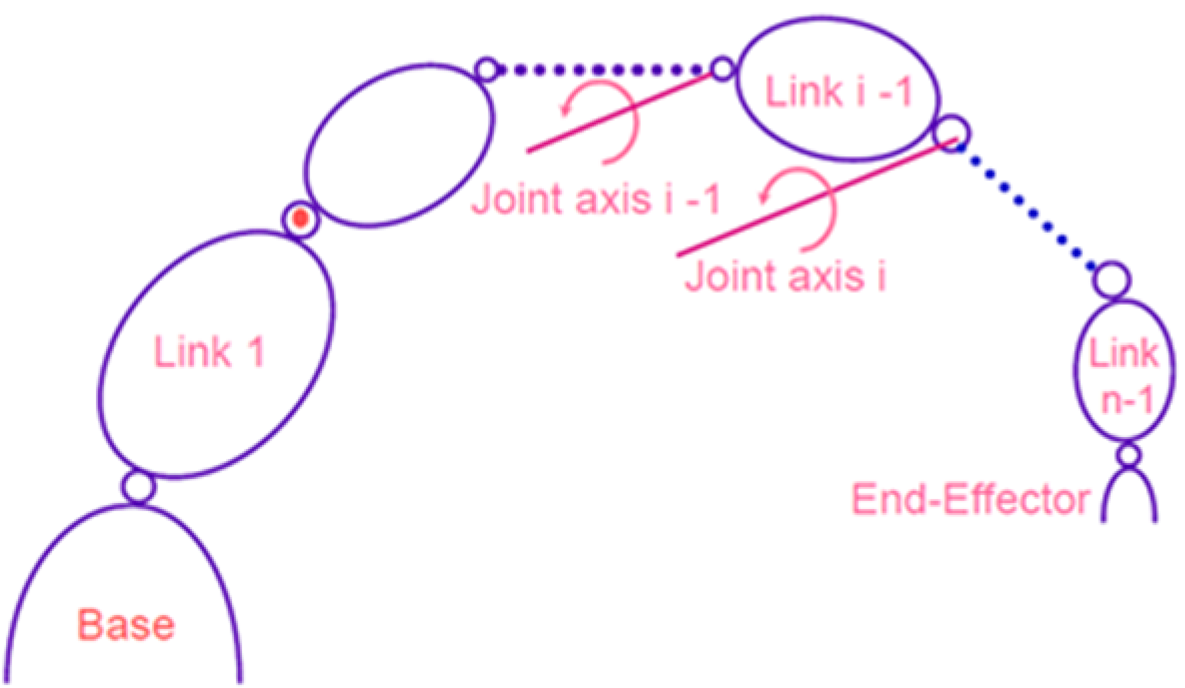
REMINDER 1: Manipulator/ Kinematic Chain

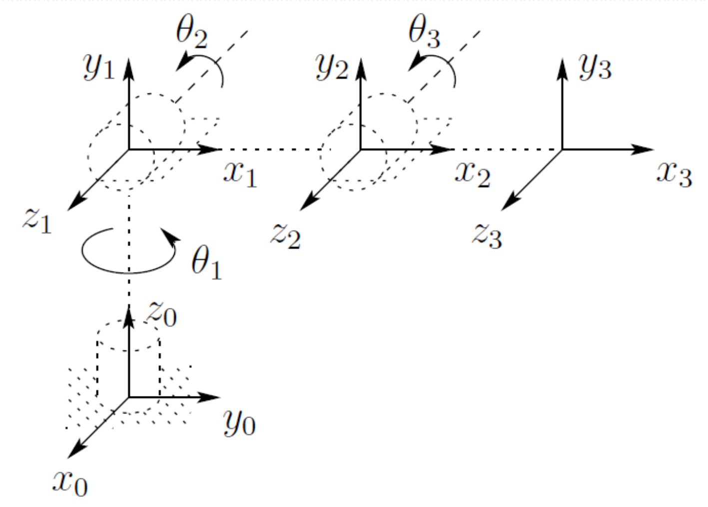


REMINDER 2: Steps of Kinematic Analysis

1. Attach frame *i* <$o_i x_i y_i z_i$> to link *i*.
   - Coordinates of points in link *i* **in frame *i*** are constant
2. Find the transformation from each frame to the next
   - Origin of frame *i* in frame *i-1*
   - $A_j = T_j^{j-1} = A_j\left(q_j\right)$
3. Find the end effector origin in the base frame
   - $T_n^0 = T_1^0 T_2^1 \ldots\ldots T_{n-1}^{n-2} T_n^{n-1}$

$$T_j^i = A_{i+1} \cdots A_j = \begin{bmatrix} R_j^i & o_j^i \\ 0 & 1 \end{bmatrix}$$

$$R_j^i = R_{i+1}^i \cdots R_j^{j-1}$$
$$o_j^i = o_{j-1}^i + R_{j-1}^i o_j^{j-1}$$



REMINDER 3: Denavit-Hartenberg Parameters

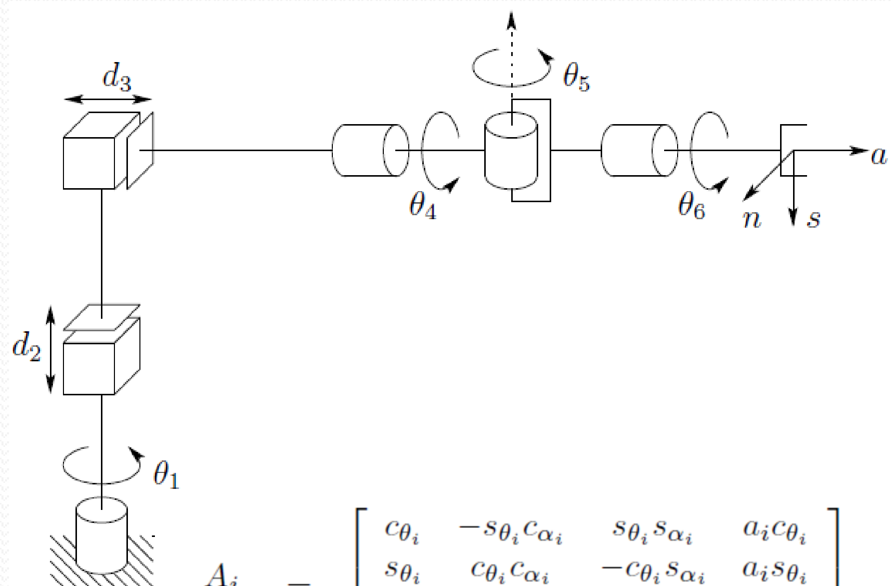
- Constants by design
  - Link twist α_i
  - Link length a_i

- Joint parameters
  - Link Offset d_i (variable in prismatic)
  - Joint Angle θ_i (variable in revolute)

- *All frame transformations are functions in these four parameters*

REMINDER 4: Cylindrical Manipulator with Spherical Wrist

Link	a_i	α_i	d_i	Θ_i
1	0	0	a_1	Θ_1
2	0	-90	d_2	0
3	0	0	d_3	0
4	0	-90	0	Θ_4
5	0	90	0	Θ_5
6	0	0	a_6	Θ_6



$$A_i = \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} c_{\alpha_i} & s_{\theta_i} s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i} c_{\alpha_i} & -c_{\theta_i} s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

a_i : distance (z_i , z_{i+1}) along x_i

α_i : angle (z_i , z_{i+1}) about x_i

d_i : distance (x_{i-1} , x_i) along z_i

θ_i : angle (x_{i-1} , x_i) about z_i

General Inverse Kinematics Problem

Given

$$H = \begin{bmatrix} R & o \\ 0 & 1 \end{bmatrix} \in SE(3)$$

Find $q_1, q_2, \ldots\ldots, q_n$

so that

$$T_n^0(q_1, \ldots, q_n) = H$$

$$T_n^0(q_1, \ldots, q_n) = A_1(q_1) \cdots A_n(q_n)$$

How to solve it?

- *12* equations in *n variables*

$$T_{ij}(q_1, \ldots, q_n) = h_{ij}, \qquad i = 1, 2, 3, \quad j = 1, \ldots, 4$$

- *Why 12?!*

Example (Stanford Arm)

$$H = \begin{bmatrix} 0 & 1 & 0 & -0.154 \\ 0 & 0 & 1 & 0.763 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned}
c_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] - s_1(s_4c_5c_6 + c_4s_6) &= 0 \\
s_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] + c_1(s_4c_5c_6 + c_4s_6) &= 0 \\
-s_2(c_4c_5c_6 - s_4s_6) - c_2s_5c_6 &= 1 \\
c_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] - s_1(-s_4c_5s_6 + c_4c_6) &= 1 \\
s_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] + c_1(-s_4c_5s_6 + c_4c_6) &= 0 \\
s_2(c_4c_5s_6 + s_4c_6) + c_2s_5s_6 &= 0 \\
c_1(c_2c_4s_5 + s_2c_5) - s_1s_4s_5 &= 0 \\
s_1(c_2c_4s_5 + s_2c_5) + c_1s_4s_5 &= 1 \\
-s_2c_4s_5 + c_2c_5 &= 0 \\
c_1s_2d_3 - s_1d_2 + d_6(c_1c_2c_4s_5 + c_1c_5s_2 - s_1s_4s_5) &= -0.154 \\
s_1s_2d_3 + c_1d_2 + d_6(c_1s_4s_5 + c_2c_4s_1s_5 + c_5s_1s_2) &= 0.763 \\
c_2d_3 + d_6(c_2c_5 - c_4s_2s_5) &= 0
\end{aligned}$$

$$\theta_1 = \pi/2$$
$$\theta_2 = \pi/2$$
$$d_3 = 0.5$$
$$\theta_4 = \pi/2$$
$$\theta_5 = 0$$
$$\theta_6 = \pi/2$$

Why Closed Form Solution

1. Fast to compute (after we discover it)

2. Finding all solutions

We find the mathematical solution then apply engineering limitations.

How to find a closed form solution

- Kinematic Decoupling

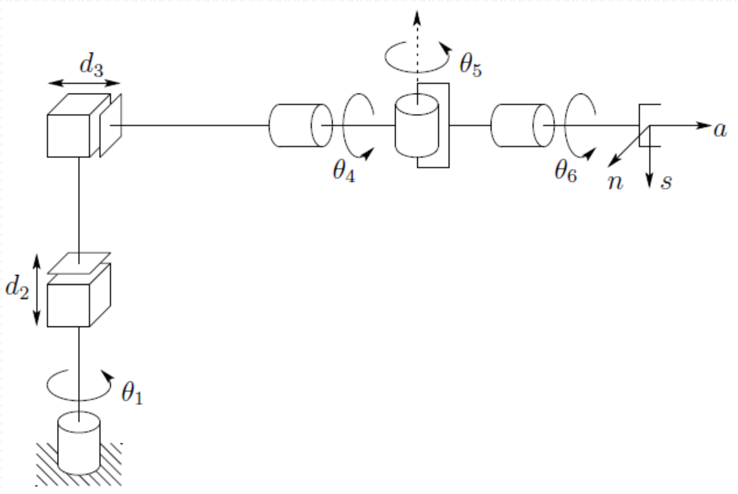
  - Inverse Position Kinematics
    - Find the parameters controlling the position

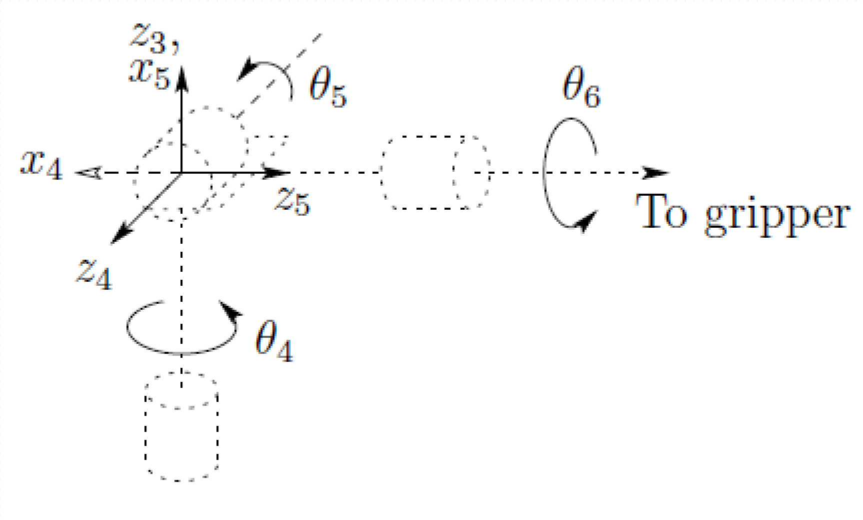
  - Inverse Orientation Kinematics
    - Find the parameters controlling the orientation

This limits possible arm designs

Most common case

- Six joints





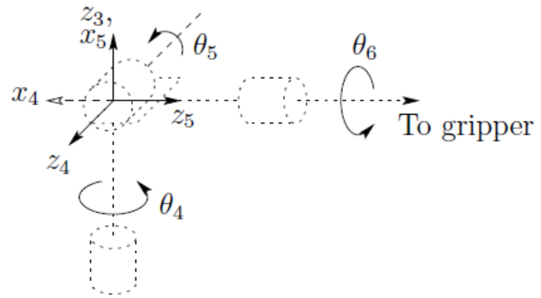
- Last three are a wrist (intersecting at some point o_c)

$$
\begin{aligned}
R_6^0(q_1, \ldots, q_6) &= R \\
o_6^0(q_1, \ldots, q_6) &= o
\end{aligned}
$$

Position of Rest Center

- The end effector is just a translation away from o_c

Link	a_i	α_i	d_i	θ_i
4	0	-90	0	θ_4^*
5	0	90	0	θ_5^*
6	0	0	d_6	θ_6^*



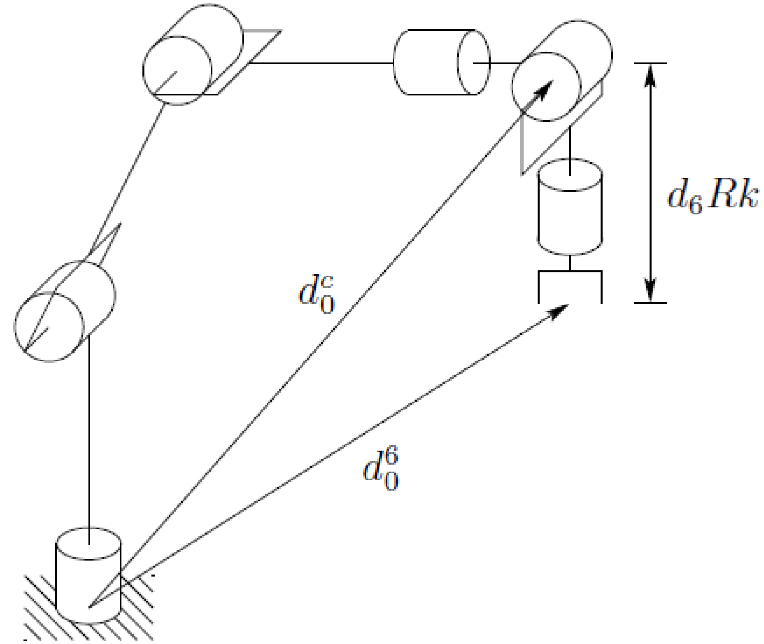
$$o = o_c^0 + d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad o_c^0 = o - d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} o_x - d_6 r_{13} \\ o_y - d_6 r_{23} \\ o_z - d_6 r_{33} \end{bmatrix}$$

This gives the first three parameters

Orientation From Wrist to Gripper

$$R = R_3^0 R_6^3$$



$$R_6^3 = (R_3^0)^{-1} R = (R_3^0)^T R$$

This gives the last three parameters

General Inverse Kinematic Approach (as we use it)

1. Using desired rotation and displacement of end effector:
   - Find the location of the wrist center
2. Using location of wrist center
   - Find first three parameters (geometrical approach)
   - *This is the solution of the inverse position problem*
3. Using the desired rotation and
   - Find last three parameters (Eular Angles)
   - *This is the solution of the inverse orientation problem*

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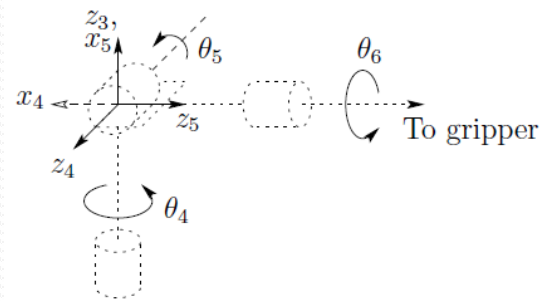

$$o = o_c^0 + d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad o_c^0 = o - d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

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This gives the first three parameters

General Inverse Kinematic Approach (as we use it)

1. ~~Using desired rotation and displacement of end effector:~~
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   - *This is the solution of the inverse orientation problem*

Inverse Position (Geometric)

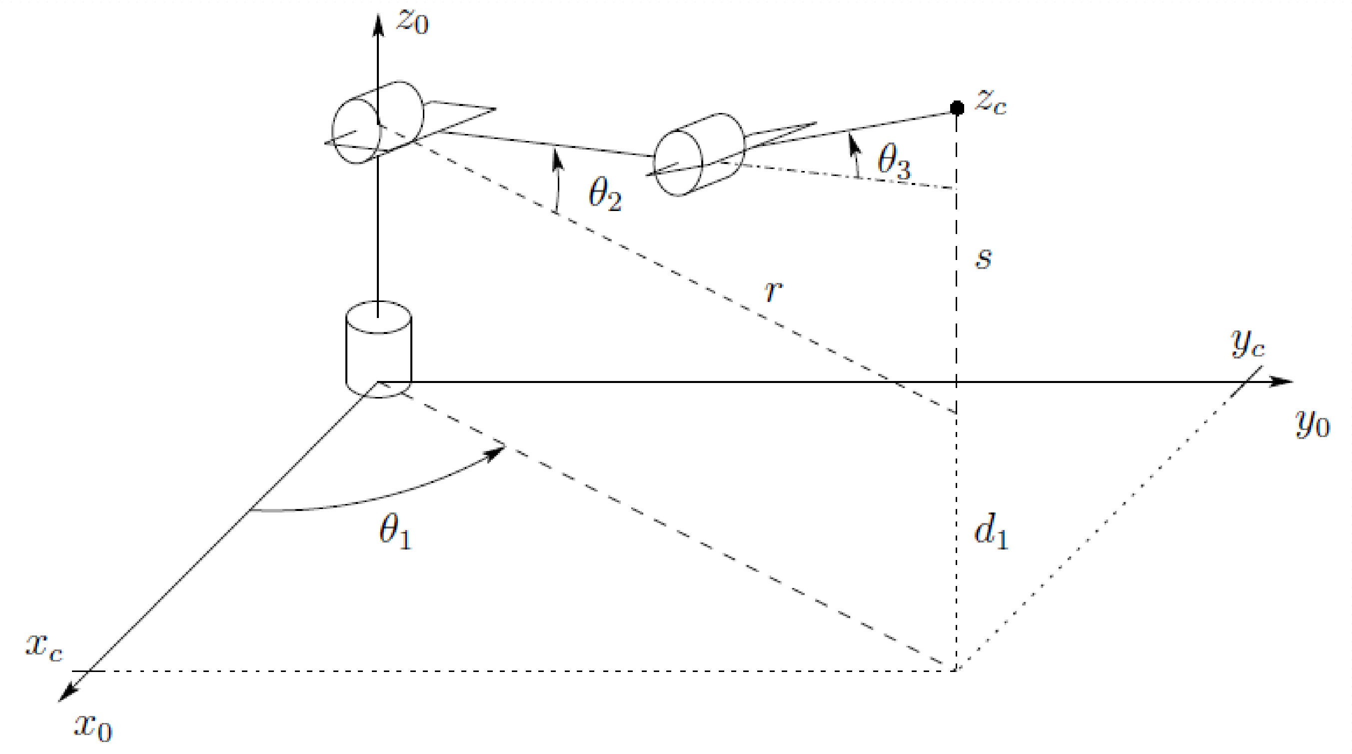
- Given o_c^0
  - Find q_1, q_2, q_3

- Steps to find q_i :
  1. Project the manipulator onto x_{i-1}-y_{i-1} plane
  2. Solve a simple trigonometry problem

- Why geometric approach?
  - Simple
  - Applies to MOST manipulators

Inverse Position Example 1

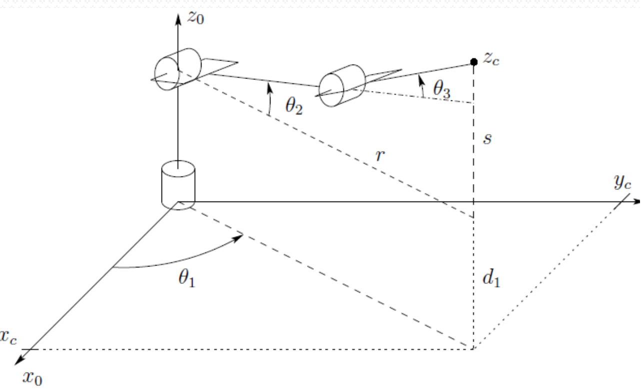
- Elbow Manipulator (e.g. PUMA)

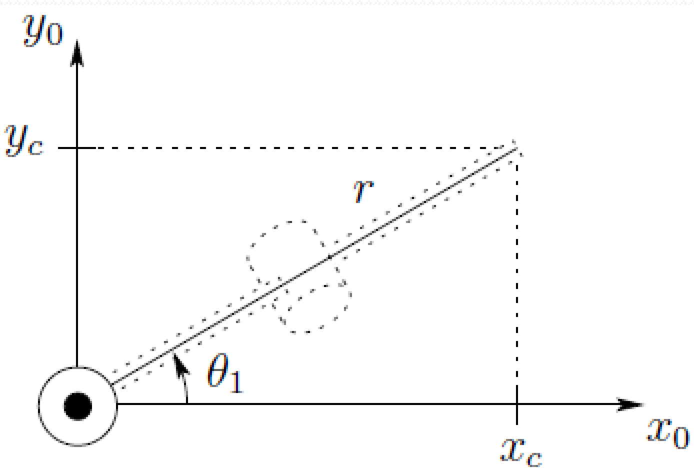


Inverse Position – Elbow Cont.

- q_1

1. Projection onto x_0-y_0





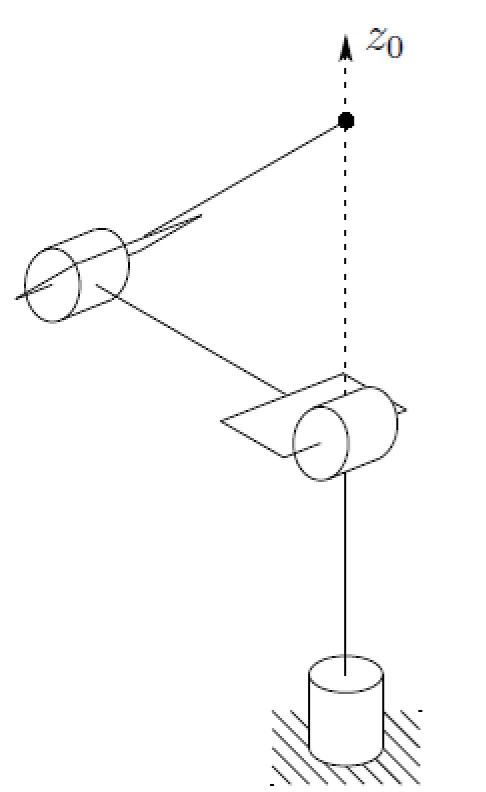
2. Solve for Θ_1

$$\theta_1 = \arctan 2(x_c, y_c)$$

or

$$\theta_1 = \pi + \arctan 2(x_c, y_c)$$

Singular Position

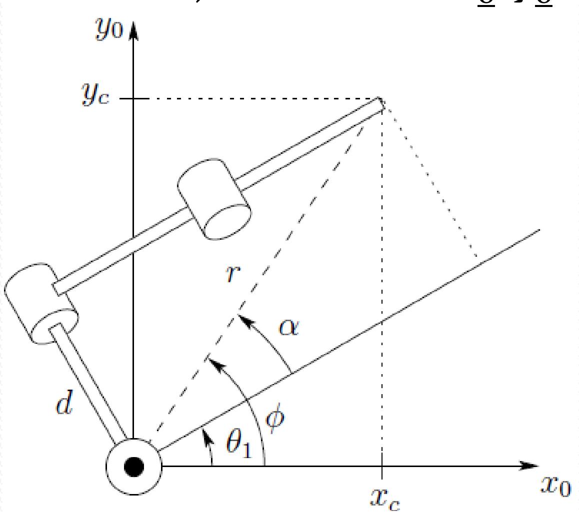


$$x_c = y_c = 0!!!!$$

Inverse Position – Elbow Cont.

- Avoiding Singularity in q_1

1. Projection onto x_0-y_0

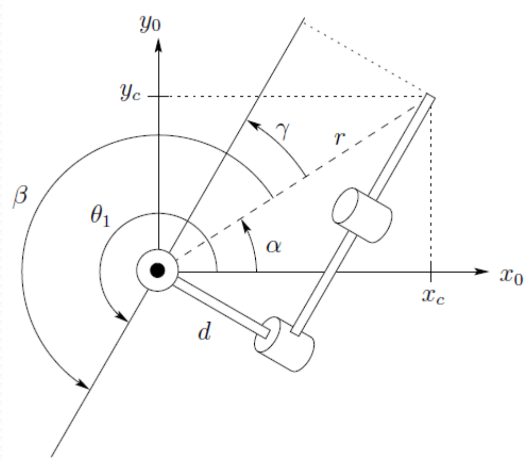


2. Solve for Θ_1

$$\theta_1 = \varphi - \alpha$$

$$\alpha = \arctan 2\left(\sqrt{r^2 - d^2}, d\right)$$

$$\varphi = \arctan 2\left(\sqrt{x^2 + y^2 - d^2}, d\right)$$

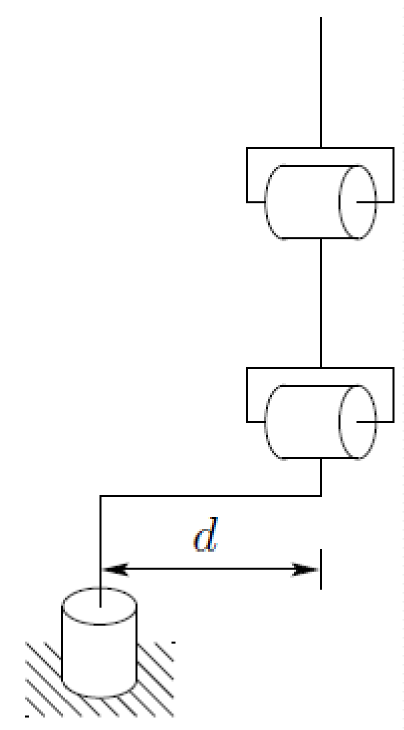


$$\theta_1 = \alpha + \beta$$

$$\alpha = \arctan 2\left(x_c, y_c\right)$$

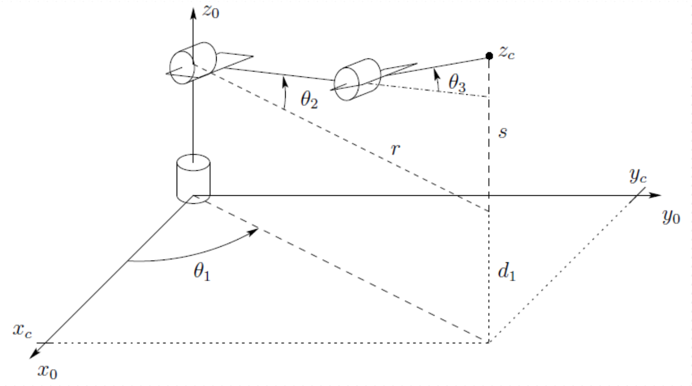
$$\beta = \gamma + \pi$$

$$\gamma = \arctan 2\left(\sqrt{r^2 - d^2}, d\right)$$

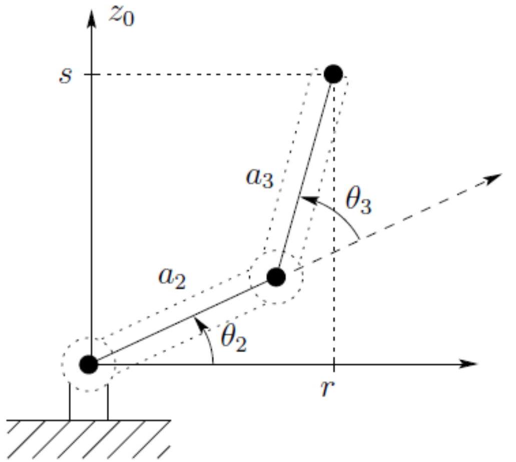


Inverse Position – Elbow Cont.

- q_2, q_3



1. Projection onto link$_2$-link$_3$ plan

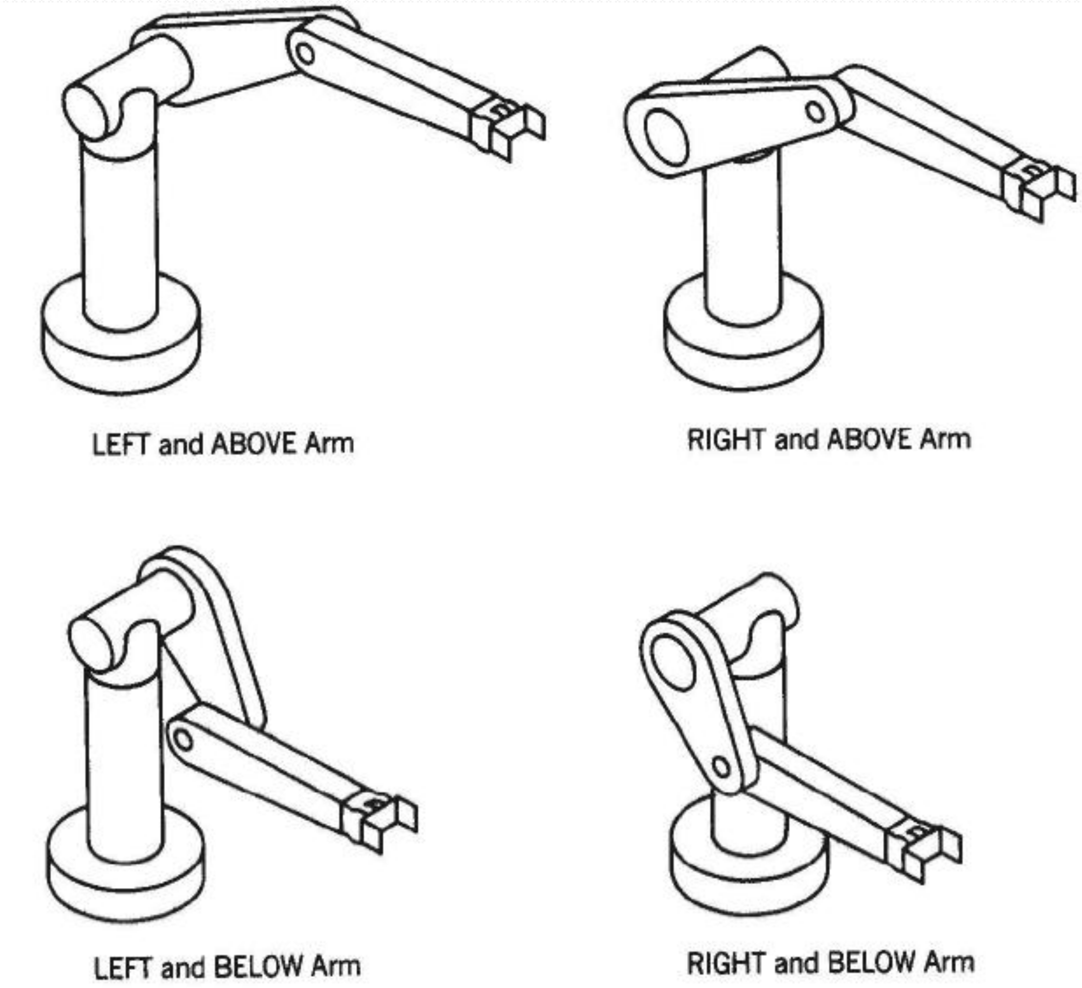


2. Solve for Θ_1

$$
\begin{aligned}
\cos\theta_3 &= \frac{r^2+s^2-a_2^2-a_3^2}{2a_2a_3} \\
&= \frac{x_c^2+y_c^2-d^2+(z_c-d_1)^2-a_2^2-a_3^2}{2a_2a_3} := D \\
&= \text{atan2}\left(D, \pm\sqrt{1-D^2}\right) \\
\theta_2 &= \text{atan2}(r,s)-\text{atan2}(a_2+a_3c_3, a_3s_3) \\
&= \text{atan2}\left(\sqrt{x_c^2+y_c^2-d^2}, z_c-d_1\right)-\text{atan2}(a_2+a_3c_3, a_3s_3)
\end{aligned}
$$

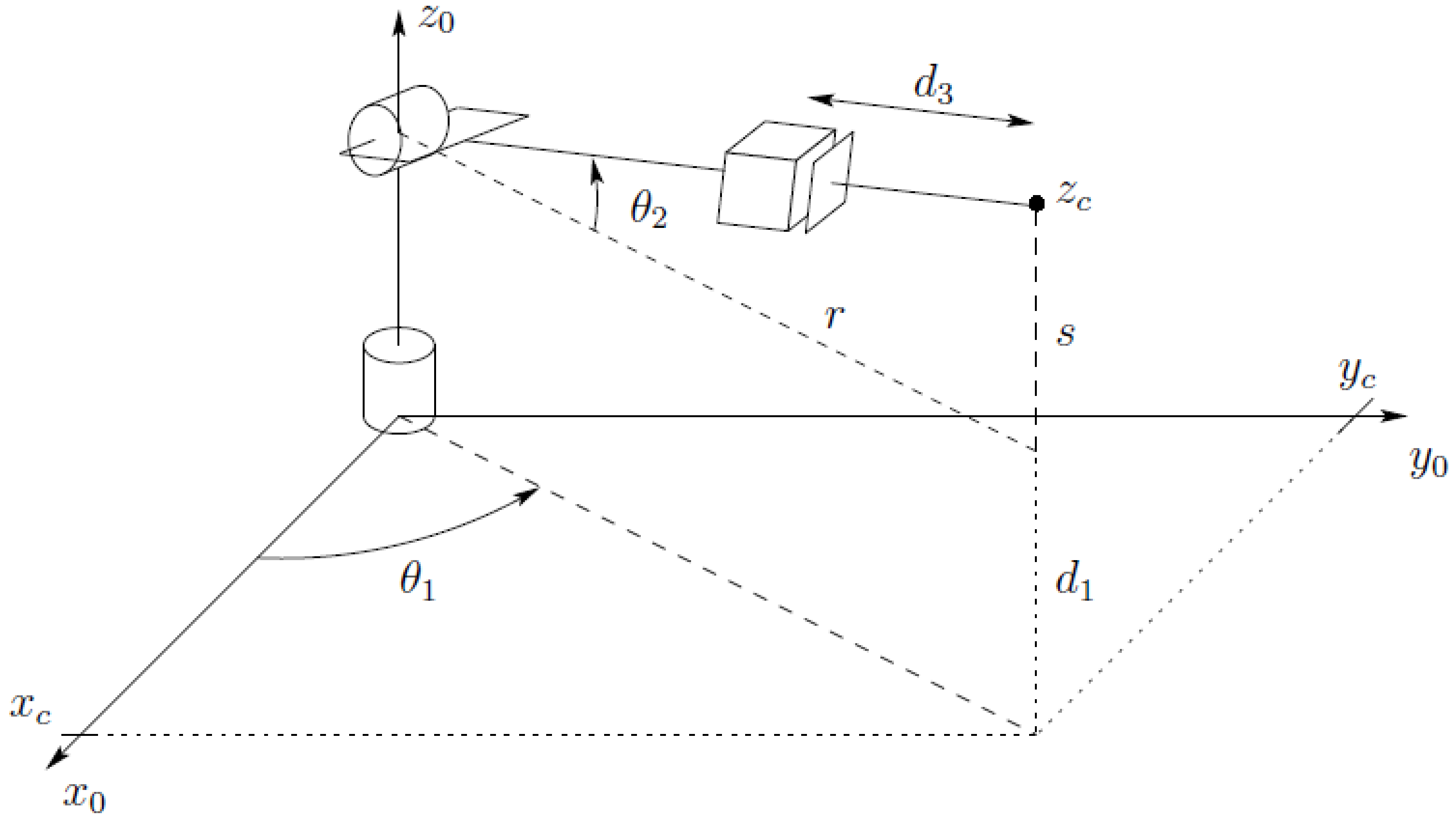
Inverse Position – Elbow Cont.

- The four solutions

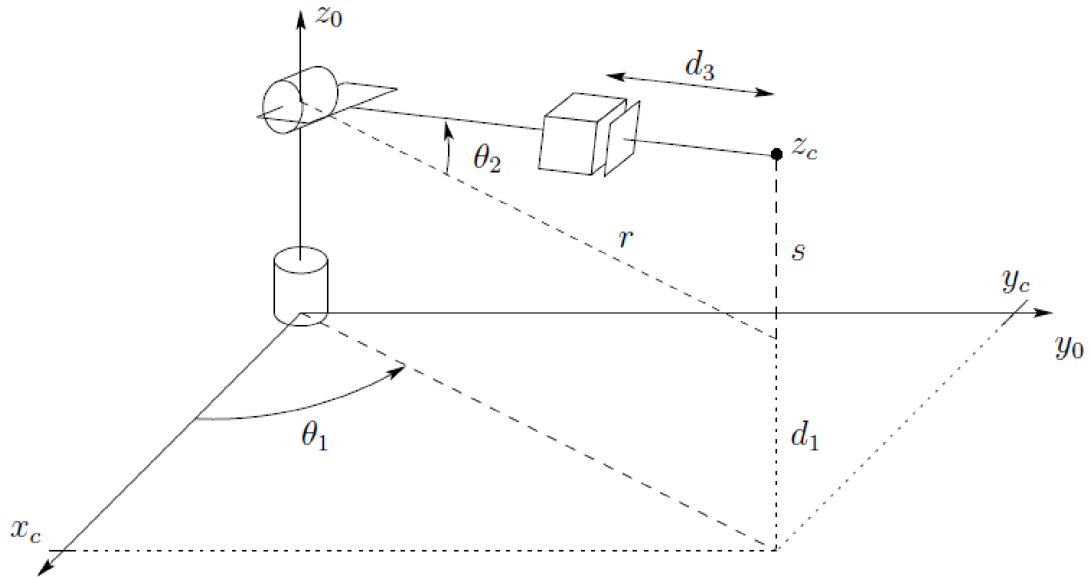


Inverse Position – Example 2

- Spherical Manipulator



Inverse Position – Spherical Cont.



$$\theta_1 = \text{atan2}(x_c, y_c)$$

or

$$\theta_1 = \pi + \text{atan2}(x_c, y_c)$$

$$\theta_2 = \text{atan2}(r, s) + \frac{\pi}{2}$$

$$d_3 = \sqrt{r^2 + s^2} = \sqrt{x_c^2 + y_c^2 + (z_c - d_1)^2}$$

General Inverse Kinematic Approach (as we use it)

1. ~~Using desired rotation and displacement of end effector:~~
   - ~~Find the location of the wrist center~~
2. ~~Using location of wrist center~~
   - ~~Find first three parameters (geometrical approach)~~
   - ~~*This is the solution of the inverse position problem*~~
3. Using the desired rotation and loc.of wrist center
   - Find last three parameters (Eular Angles)
   - *This is the solution of the inverse orientation problem*

Inverse Orientation

Using the desired rotation and loc.of wrist center

Find last three parameters

Can be interpreted as finding Eular Angles

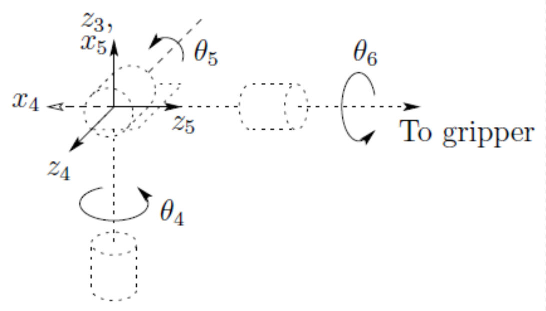
$$\theta_4 = \phi$$

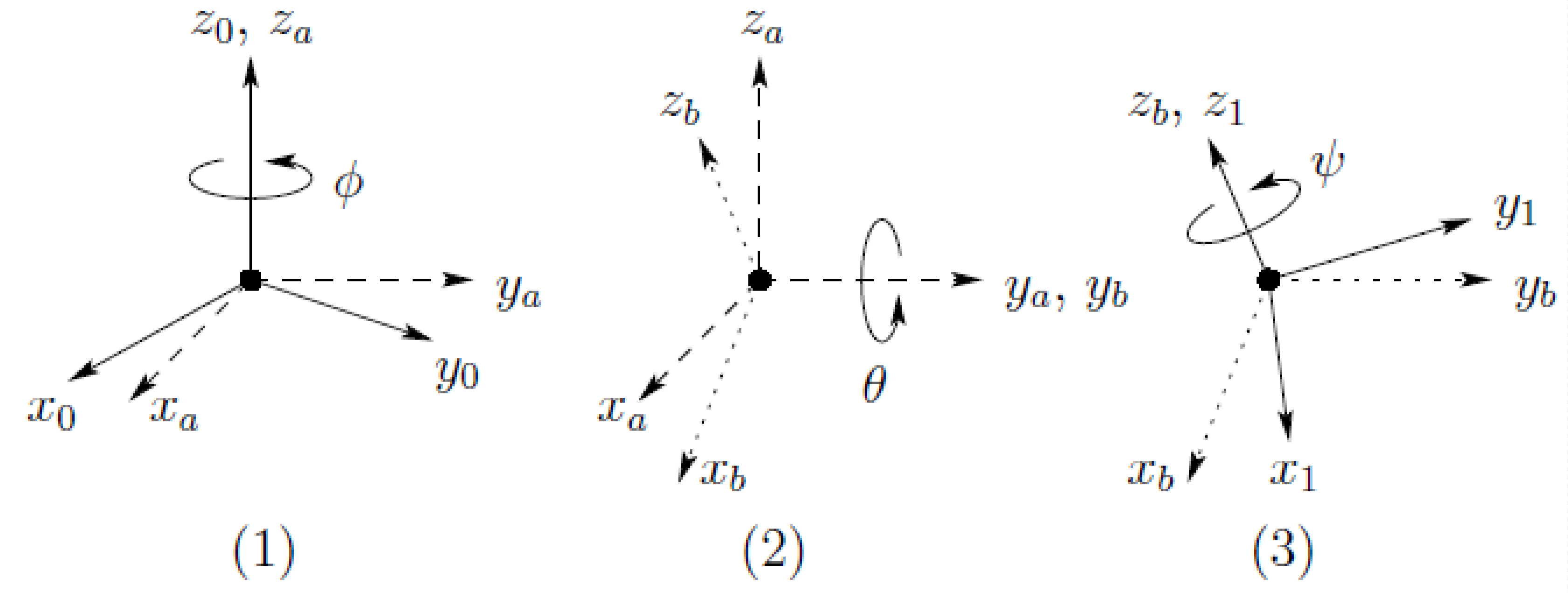
$$\theta_5 = \theta$$

$$\theta_6 = \psi$$

Eular Angles

- Φ, Θ, Ψ
- R_{ZYZ}





From R to Eular Angles

- Given R
  - Find Φ, Θ, Ψ

$$\begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

- If r_{13} and r_{23} are not both zero
  - $s_\Theta \neq 0$, r_{31} and r_{32} are not both zero, $r_{33} \neq \pm 1$, $c_\Theta = r_{33}$, $s_\Theta = \pm\sqrt{1 - r_{33}}$

$$\begin{aligned} \theta &= \operatorname{atan2}\left(r_{33}, \sqrt{1 - r_{33}^2}\right) \\ \phi &= \operatorname{atan2}(r_{13}, r_{23}) \\ \psi &= \operatorname{atan2}(-r_{31}, r_{32}) \end{aligned}$$

$$\begin{aligned} \theta &= \operatorname{atan2}\left(r_{33}, -\sqrt{1 - r_{33}^2}\right) \\ \phi &= \operatorname{atan2}(-r_{13}, -r_{23}) \\ \psi &= \operatorname{atan2}(r_{31}, -r_{32}) \end{aligned}$$

From R to Eular Angles 2

- Given R
  - Find Φ, Θ, Ψ

$$\begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

- If r_{13} and r_{23} are both zeros
  - $s_\Theta=0$, $r_{31} = r_{32} = 0$, $r_{33} = \pm 1$, $c_\Theta = 1$, $\Theta = 0$ *(Two rotations around Z)*

$$\begin{bmatrix} c_\phi c_\psi - s_\phi s_\psi & -c_\phi s_\psi - s_\phi c_\psi & 0 \\ s_\phi c_\psi + c_\phi s_\psi & -s_\phi s_\psi + c_\phi c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_{\phi+\psi} & -s_{\phi+\psi} & 0 \\ s_{\phi+\psi} & c_{\phi+\psi} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

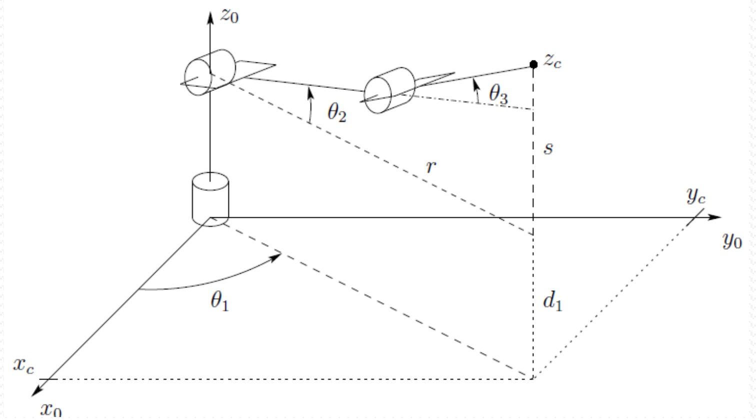
$$\begin{aligned} \phi + \psi &= \operatorname{atan2}(r_{11}, r_{21}) \\ &= \operatorname{atan2}(r_{11}, -r_{12}) \end{aligned}$$

$$\begin{bmatrix} -c_{\phi-\psi} & -s_{\phi-\psi} & 0 \\ s_{\phi-\psi} & c_{\phi-\psi} & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\phi - \psi = \operatorname{atan2}(-r_{11}, -r_{12})$$

Elbow Manipulator – Full Solution



Given

$$o = \begin{bmatrix} o_x \\ o_y \\ o_z \end{bmatrix}; \quad R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\begin{aligned} x_c &= o_x - d_6 r_{13} \\ y_c &= o_y - d_6 r_{23} \\ z_c &= o_z - d_6 r_{33} \end{aligned}$$

$$\begin{aligned}
\theta_1 &= \text{atan2}(x_c, y_c) \\
\theta_2 &= \text{atan2}\left(\sqrt{x_c^2 + y_c^2 - d^2}, z_c - d_1\right) - \text{atan2}(a_2 + a_3 c_3, a_3 s_3) \\
\theta_3 &= \text{atan2}\left(D, \pm\sqrt{1 - D^2}\right), \\
& \qquad \textit{where } D = \frac{x_c^2 + y_c^2 - d^2 + (z_c - d_1)^2 - a_2^2 - a_3^2}{2 a_2 a_3} \\
\theta_4 &= \text{atan2}(c_1 c_{23} r_{13} + s_1 c_{23} r_{23} + s_{23} r_{33}, \\
& \qquad -c_1 s_{23} r_{13} - s_1 s_{23} r_{23} + c_{23} r_{33}) \\
\theta_5 &= \text{atan2}\left(s_1 r_{13} - c_1 r_{23}, \pm\sqrt{1 - (s_1 r_{13} - c_1 r_{23})^2}\right) \\
\theta_6 &= \text{atan2}(-s_1 r_{11} + c_1 r_{21}, s_1 r_{12} - c_1 r_{22})
\end{aligned}$$